Joint Precoder and Combiner Design for MMSE Distributed Beamforming with Per-Antenna Power Constraints

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Abstract—We consider minimum mean square error (MMSE) joint precoder and combiner design for single and multi-carrier distributed beamforming systems with nonuniform per-antenna transmit power constraints. We show that, similar to the maximum-gain problem, an iterative Gauss-Seidel algorithm can be used for minimizing MSE which alternately optimizes the transmitter and receiver coefficients. In a single carrier system the optimum transmit coefficients are obtained by a simple projection of the effective MISO channel. In the multicarrier case with a sum-MSE objective, the Gauss-Seidel approach is once again applicable, but the transmit coefficients must be found by solving a quadratically constrained quadratic problem for which we apply a dual gradient algorithm. A numerical example is presented which shows improvement of 0.7 dB in carrier signal-to-noise ratio (SNR) relative to a projected eigenvector method for a multicarrier DBF system with Rayleigh-faded multipath channels.

Index Terms—distributd beamforming, per-antenna power constraints (PAPC), precoder design

I. INTRODUCTION

Distributed beamforming (DBF), is a promising technology for increasing communication range, providing improved signal-to-noise ratio (SNR), or reducing probability of intercept and detection for geographically separated groups of communicating nodes [1]. In a DBF system, multiple transmitters with potentially heterogeneous hardware form an antenna array in an ad-hoc manner. Each transmit radio is power constrained and since the group is heterogeneous, this imposes a set of nonuniform per-antenna power constraints (PAPC) on the transmit array. On the receiving end, another group of radios forms a receive array with nonuniform noise profile, again due to heterogeneous hardware.

Several studies of multiple input multiple output (MIMO) communications with uniform and nonuniform PAPC have appeared recently for various scenarios and objectives. For example, in [2] the authors consider precoder design for a multiple input single output (MISO) system with an outage probability objective. In [3] a zero-forcing precoder is designed for a broadcast channel with sum-rate objective. In [4], transmit beamformers are designed for a multiuser scenario with a signal-to-leakage-plus-noise criterion. In [5] multiuser sum rate is maximized, again for MISO channels. In [6] directional beamforming under PAPC is considered for MISO channels. In [7] a downlink cellular max-min problem is formulated for SINR over a set of users. Joint transmit and receive beamforming optimization with PAPC has also been studied. Optimum precoders (in terms of beamforming gain) for various combining strategies with uniform PAPC, also called equal gain transmission (EGT) were presented in [8]. Zheng, et. al. [9] then proposed a cyclic (Gauss-Seidel) algorithm for joint optimization of the precoder and combiner weights, again with a gain objective. Since then, the Gauss-Seidel approach has been applied for multiuser MIMO cellular downlinks with sum-MSE objectives in [10] and for multicarrier systems with arithmetic error probability objectives in [11]. In this paper we consider joint precoder and combiner optimization for single and multicarrier beamforming systems with PAPC and MSE objectives.

It is well known that in a narrowband MIMO beamforming system (i.e., where the precoder and combiner matrices consist of only one column), maximum gain is achieved when the precoder and combiner weight vectors lie in the directions of the channel’s dominant right and left eigenvectors, respectively. This principle is used to guide beamforming weight selection with a total-transmit-power (across all antennas) constraint. However, when per-antenna constraints are imposed, the gain-maximizing transmit and receive weights cannot be found in closed form. Instead one can appeal to the Gauss-Seidel approach which involves alternately optimizing the transmitter and receiver weights until convergence [9]–[12]. With EGT constraints, and max-gain objective, the transmit weights are found by projecting the effective MISO channel onto the set of vectors with unit-magnitude components [8], while the receive weights satisfy the maximum ratio combining (MRC) principle and lie in the direction of the effective single input multiple output (SIMO) channel.

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In this paper, we show that a Gauss-Seidel approach can be applied to find the MMSE-optimal transmit and receive weights for both narrowband and wideband (multicarrier) systems. We make the following contributions:

1) We show that, similar to the maximum gain case, the single-carrier MMSE transmit weights under PAPC are obtained by a simple projection of the MISO channel but only when this channel satisfies a certain norm condition. We then show that a Gauss-Seidel algorithm can be used to jointly optimize the transmit and receive weights for MMSE under PAPC.

2) We find the optimal transmit weights with PAPC for a multicarrier system with a sum-MSE objective using a dual gradient algorithm which we then embed in a Gauss-Seidel algorithm to find the jointly optimal transmit and receive weights with PAPC.

We use the following notation. Boldfaced lowercase symbols represent complex vectors and boldfaced uppercase symbols represent complex matrices. The superscripts $^T$ and $^H$ denote transpose and conjugate (Hermitian) transpose of a matrix or vector, respectively and $^*$ indicates the complex conjugate. The quantity $\text{diag}(a_1, \ldots, a_n)$ is a diagonal matrix with $a_1, \ldots, a_n$ on the diagonal and $\text{blkdiag}(A_1, \ldots, A_n)$ is a block-diagonal matrix with $A_1, \ldots, A_n$ on the diagonal. Statistical expectation is denoted by $\mathbb{E}[:]$. The magnitude (modulus) of a complex number $a$ is denoted $|a|$ and the angle by $\angle a$. Vector norms are represented by $\|\cdot\|_2$ with the subscript denoting the type of norm, and projection operators are denoted $[\cdot]^+$ with the superscript indicating the projection set. Finally, $\mathbb{R}_+^n$ and $\mathbb{R}^n_-$ are the positive and non-negative orthants of $\mathbb{R}^n$, respectively.

The remainder of the paper is organized as follows. Section II presents the MIMO beamforming system model and introduces the projection and norm operators which facilitate the analysis in the sequel. Section III summarizes the max-gain problem which provides useful comparisons to the MMSE systems which we develop later. Next in Section IV we derive the jointly optimal MMSE transmit and receive weights for single carrier systems and Section V extends this to multicarrier systems with sum-MSE objective. A numerical example is presented in Section VI which shows the utility of the proposed algorithms. Finally, the paper is concluded in Section VII.

II. SYSTEM MODEL

A. PAPC MIMO Beamforming System

The $n$-transmitter, $m$-receiver MIMO beamforming system is described by the following equation

$$\hat{s} = w^H H z s + w^H n$$

(1)

where $s$ and $\hat{s}$ are the complex transmitted and equalized symbols, respectively, $n \in \mathbb{C}^m$ is the receive noise vector, $H$ is the $m \times n$ complex MIMO channel matrix, and $z \in \mathbb{C}^n$ and $w \in \mathbb{C}^m$ are the transmit and receive beamforming weight vectors, respectively. The multicarrier case is modeled by

$$\hat{s}_k = w_k^H H_k z_k s_k + w_k^H n_k$$

where the subscript $k$ indicates the $k$th carrier. We assume a distributed system in which some node, which we call the fusion center in keeping with standard DBF terminology, has full channel state information (CSI). The fusion center computes the jointly optimal transmit and receive weight vectors and feeds this information to the other nodes. We consider per-antenna transmitter power constraints of the form

$$|z_i| \leq \sqrt{p_i}, \; i = 1, \ldots, n$$

(2)

where $z_i$ is the $i$th component of the transmit weight vector and $p_i$ is the maximum allowable power of the $i$th element. In the multicarrier case, we constrain the total power across carriers: $\sum_k |z_{i,k}|^2 \leq p_i$ where $z_{i,k}$ is the transmit weight of the $k$th carrier on antenna $i$.

B. $P$-Projection and $P$-Norm

Define the constraint vector $p = [p_1, \ldots, p_n]^T$ and matrix $P = \text{diag}(p_1, \ldots, p_n)$ and the total power $p_T = \sum_i p_i$. Define $P$ as the set of feasible transmit vectors which meet the constraints (2) with equality (i.e., the boundary of the feasible set): $P = \{ z \in \mathbb{C}^n : |z_i| = \sqrt{p_i}, \; i = 1, \ldots, n \}$.

It can be easily shown that the closest $z \in P$ to an arbitrary nonzero $x = [x_1, \ldots, x_n]^T \in \mathbb{C}^n$ is

$$[x]^P = \underset{z \in P}{\text{argmin}} \|x - z\|_2 = \left[ \sqrt{p_1} e^{j \angle x_1}, \ldots, \sqrt{p_n} e^{j \angle x_n} \right]^T$$

(3)

and thus $[x]^P$ is a projection operator onto the set $P$ [13], which we call the $P$-projection. Equation (3) is valid for any $\ell_2$ norm as well as the following “weighted” $\ell_1$ norm [14], which we call the $P$-norm, and which will facilitate further analysis.

$$\|x\|_P = \|P^{1/2} x\|_1 = \sum_{i=1}^n \sqrt{p_i} |x_i|$$

(4)

We note a few important facts regarding the $P$-projection and the $P$-norm. First, when $x \in P$, its norm is $\|x\|_P = p_T$. When the power constraints are all unity, i.e., $P = I$, the $P$-norm is equivalent to the $\ell_1$ norm. From (3), the $P$-projection is independent of the magnitudes of the components of $x$ and thus $[x]^P = [\alpha x]^P$ for any nonzero real $\alpha$. As with any norm, we have $|\alpha x|_P = |\alpha| \cdot \|x\|_P$ for any real $\alpha$. Finally, for any nonzero $x$, we have $x^H [x]^P = \|x\|_P^2$.

C. Constrained MMSE Problems

In this paper, our goal is to minimize MSE subject to PAPC. From the narrowband model (1). The normalized MSE is given by

$$\xi(z, w; H, R_n) = \frac{1}{\sigma_s^2} \mathbb{E}[|\hat{s} - s|^2] = |w^H H z - 1|^2 + \frac{1}{\sigma_s^2} w^H R_n w$$

(4)
where $\sigma_z^2 = E[|z|^2]$ and $R_n = E[nn^H]$ is the noise covariance. When $\sigma_z^2 = 1$, the transmit power of the $i$th antenna is $|z_i|^2$ and we will make this assumption henceforth. The single-carrier constrained MMSE problem is

$$\begin{align*}
\minimize_{z \in \mathbb{C}^n, w \in \mathbb{C}^m} & \quad \xi(z, w; H, R_n) \\
\text{subject to} & \quad |z_i| \leq \sqrt{p_i}, \quad i = 1, \ldots, n.
\end{align*}$$

(5)

With multiple carriers, we minimize the sum-MSE across carriers and constrain the total power at each antenna. Sum-MSE has been used as an objective for multi-user systems [10], spatial multiplexing systems [15], and multicarrier systems [16]. The joint optimization problem is

$$\begin{align*}
\minimize_{z_1, \ldots, z_K, w_1, \ldots, w_K \in \mathbb{C}^n} & \quad \sum_{k=1}^K \xi(z_k, w_k; H_k, R_{n,k}) \\
\text{subject to} & \quad \sum_k |z_{i,k}|^2 \leq p_i, \quad i = 1, \ldots, n.
\end{align*}$$

(6)

III. MAX GAIN BEAMFORMING WITH PAPC

The problem of maximizing the gain $G = |w^HHz|^2$ under EGT constraints has been studied extensively, e.g., [8], [9] and the focus of the present paper is MSE minimization. However, a brief summary of the results on gain maximization is in order here as there are many parallels to the MMSE case. Thus, in this section we provide this summary and in so doing, we also extend the EGT results to the case of nonuniform PAPC.

A. Max-Gain Precoder and Combiner

The optimum unit-norm receive weight $w$ with transmit weight $z$ is the MRC vector

$$w = \frac{Hz}{\|Hz\|_2} e^{j\phi}$$

(7)

where $\phi$ is an arbitrary phase shift. With no individual transmit power constraints, the optimum (maximal ratio transmission) unit-norm transmit weight vector for receive weight $w$ is

$$z = \frac{H^Hw}{\|H^Hw\|_2} e^{j\theta}$$

(8)

where $\theta$ is again arbitrary. The jointly optimal weights are $w = u_1 e^{j\phi}$, $z = v_1 e^{j\theta}$ where $u_1$ and $v_1$ are the dominant left and right eigenvectors of $H$.

B. Max Gain Precoder and Combiner with PAPC

With PAPC on the transmitter, the optimum transmit vector $z$ with receive weight vector $w$ is

$$z = [H^Hw]^p e^{j\theta}$$

(9)

with arbitrary $\theta$. Note that the optimum EGT weight vector given in [8] is a special case of the above equation, with $p = 1$. Assuming MRC receive weights, the optimum transmit phase vector $\psi = [\angle z_1, \ldots, \angle z_n]^T$ solves $\psi^* = \arg\max_{\psi} \|H^1/2e^{j\psi}w\|_2$ (also a generalization of a result from [8]).

C. Joint Gain Maximization with Gauss-Seidel Algorithm

While a closed-form solution for the jointly optimal weights has not been found, [9] proposed a Gauss-Seidel (or “cyclic”) algorithm wherein the transmit and receive weights are alternately updated according to (7) and (9). Gauss-Seidel algorithms are also referred to as block coordinate descent algorithms and their convergence is studied in [17] and [18]. They are appealing since at every iteration, the objective is guaranteed to improve or stay the same. However, convergence can only be to a local minimum (for objectives which are non-convex in the joint set of decision variables), and even this convergence requires somewhat restrictive conditions [10], [17]. Nevertheless, the Gauss-Seidel approach has been applied successfully in several MIMO precoder/combiner optimization problems [9]–[11]. In the max-gain case, the algorithm starts with initializing $w^{(0)}$, for example to $u_1$. Then equations (9) and (7) are alternately applied until convergence.

IV. SINGLE-CARRIER MMSE BEAMFORMING WITH PAPC

We now turn to the MMSE problem, which is the focus of this paper. In this section we study the single-carrier case in depth and in the next section we study the multicarrier case. Our goal is to minimize the MSE subject to PAPC, i.e., to solve problem (5). Note that the MSE (4) is convex in $z$ for a fixed $w$ and vice versa, but it is not convex in $(z, w)$ [16], which provides some motivation for a Gauss-Seidel approach. We first summarize the MMSE problem without PAPC.

A. MMSE Precoder and Combiner

The MMSE receive weight $w$ for a fixed transmit weight vector $z$ is

$$w = \frac{R_n^{-1}Hz}{1 + z^H\hat{H}R_n^{-1}\hat{H}z}.$$  

(10)

The optimal transmit weight vector for fixed $w$ is

$$z = \frac{\hat{H}w}{\|\hat{H}w\|_2^2}$$

which is the same as (8) up to a constant. Joint optimization of $w$ and $z$ is accomplished by expressing the resultant MSE with the optimal receive vector (10) as

$$\xi(z, H, R_n) = \frac{1}{1 + z^H\hat{H}R_n^{-1}\hat{H}z}$$

(11)

which is minimized for a maximum total transmit power $p_T$ by $z^* = \sqrt{p_T} \zeta$ where $\zeta$ is the (unit norm) dominant eigenvector of $H^H\hat{H}R_n^{-1}H$ [16].

B. MMSE Precoder and Combiner with PAPC

1) Constrained Problem: To find the jointly optimal transmit and receive weights for the MMSE problem with PAPC (5) we could try to maximize the denominator of the RHS of (11) under per-antenna power constraints, but the problem would not be convex. Instead we attempt to find a formula analogous to (9) which optimizes the transmitter weights $z$ for fixed receive weights $w$ with MSE objective. This, along with (10) can then be used in a cyclic algorithm to jointly
optimize \( w \) and \( z \) for MSE. From (4), we see that minimizing 
\( \xi(z, w; H, R_n) \) with respect to \( z \) is equivalent to minimizing 
\( |z^Hg|^2 - 2\text{Re}(z^Hg) \), where 
\begin{equation}
\begin{aligned}
g &= H^Hw \\
&= \text{H}^Hw \\
\end{aligned}
\end{equation}
is the effective MISO channel. The transmitter optimization problem is

\begin{equation}
\begin{aligned}
\text{minimize}_{\zeta \in \mathbb{C}^n} & \quad z^H G z - 2 \text{Re}(z^H g) \\
\text{subject to} & \quad z^H E_i z \leq p_i, \quad i = 1, \ldots, n \\
\end{aligned}
\end{equation}

where \( G = gg^H \), \( E_i = e_i e_i^T \) and \( e_i \) is the \( i \)th standard basis vector of \( \mathbb{R}^n \). In Appendix A we show that this quadratically constrained quadratic problem (QCQP) exhibits strong duality. The solutions are given in the following three propositions, which are proved in Appendix A. These solutions depend on the MISO channel \( g \) given in (12).

\textbf{Proposition 1:} If \( ||g||_p \leq 1 \), then \( z^* = |g|^p \) is an optimal power-constrained transmit beamforming weight vector, i.e., a solution to (13), with non-negative Lagrange multipliers \( \lambda_i^* = |g_i|^{1/2} - (1 - ||g||_p) \), \( i = 1, \ldots, n \).

\textbf{Proposition 2:} If \( \min_{|g|} |g| \geq \left( \sum_k \sqrt{p_k} \right)^{-1} \), then \( z^* \) is a solution to (13) with Lagrange multipliers \( \lambda^* = 0 \), where \( z_i^* = \left( \sum_k \sqrt{p_k} \right)^{-1} \).

\textbf{Proposition 3:} If \( |g| \geq n^{-1} p_i^{-1/2} \) for all \( i = 1, \ldots, n \), then \( z^* \) is a solution to (13) with Lagrange multipliers \( \lambda^* = 0 \), where \( z_i^* = \left( \sum_k \sqrt{p_k} \right)^{-1} \).

Of the solutions above, only the first (Proposition 1) has active constraints. This solution is also nearly identical to the max-gain under PAPC solution (9). However this solution is only valid when the \( \mathcal{P} \)-norm of the effective MISO channel \( g = H^Hw \) is less than unity. We show next, however, that in the context of a Gauss-Seidel algorithm for joint optimization of \( z \) and \( w \), this condition is easily managed and the solution of Proposition 1 can be used throughout the algorithm’s iterations.

\textbf{Proposition 4:} If \( ||H^Hw||_p > 1 \), then \( z^* = [H^Hw]^p \) is the optimal power-constrained transmit beamforming vector for receive vector 
\begin{equation}
\begin{aligned}
\hat{w} &= \frac{w}{||H^Hw||_p} \\
\text{and the MSE with } z^* \text{ and } \hat{w} \text{ is lower than any achievable MSE with receive vector } w. \text{ That is } \xi(z^*, \hat{w}; H, R_n) < \xi(z, w; H, R_n), \forall z \in \mathbb{C}^n.
\end{aligned}
\end{equation}

\textbf{Proof.} Assume \( ||H^Hw||_p > 1 \) and define \( z^* \) and \( \hat{w} \) as above. Since \( ||H^Hw||_p = 1 \), and \( z^* = [H^Hw]^p = [H^H\hat{w}]^p \), by Proposition 1, \( z^* \) is optimal for \( \hat{w} \). Recall that \( g^H[H^Hw]^p = ||g||_p \) for any \( g \). Thus we have \( \hat{w}^HZ^* = (H^Hw)^p/[H^Hw]^p = ||H^Hw||_p \) is 1. Therefore, from (4)
\begin{equation}
\begin{aligned}
\xi(z, w; H, R_n) &= |w^HZ - 1|^2 + w^HR_nw \\
\xi(z^*, \hat{w}; H, R_n) &= 0 + \frac{w^HR_nw}{||H^Hw||_p^2}.
\end{aligned}
\end{equation}

\square

2) \textbf{Joint Transmit/Receive Optimization:} From Proposition 4, if \( ||H^Hw||_p > 1 \), we know that the jointly optimal solution cannot include \( w \). From Proposition 1, if \( ||H^Hw||_p \leq 1 \), \( z = [H^Hw]^p \) is optimal for \( w \). Thus the optimal pair must satisfy \( ||H^Hw||_p \leq 1 \), and \( z = [H^Hw]^p \). Furthermore, when \( z \) is equal to the \( \mathcal{P} \)-projection, we have \( H^HHz = ||H^Hw||_p \). Thus the joint optimization problem is

\begin{equation}
\begin{aligned}
\text{minimize} & \quad (1 - ||H^Hw||_p)^2 + w^HR_nw \\
\text{subject to} & \quad ||H^Hw||_p \leq 1.
\end{aligned}
\end{equation}

This problem is non-convex but has a simple solution in the MISO case \((m = 1)\), with receiver noise variance \( \sigma_n^2 \), \( n \times 1 \) channel \( h \), and scalar receive weight \( w \in \mathbb{C} \). The MISO problem is independent of \( \angle w \) and convex in \( |w| \) with solution 
\begin{equation}
\begin{aligned}
w^* &= \frac{||h||_p}{\sigma_n^2 + ||h||_p^2} e^{j\phi}, \quad z^* = [h^*e^{j\phi}]^p
\end{aligned}
\end{equation}

with \( \phi \) arbitrary.

3) \textbf{Shadow Prices:} The resultant MISO with \( z = [g]^p \) is given above in (14) and is differentiable as a function of \( p_i \) with \( \partial \xi / \partial p_i = -\lambda_i \). The “shadow price” interpretation of \( \lambda_i \) is the reduction in MSE that can be realized per unit of power relaxation of the \( i \)th power constraint (for small relaxations) [13]. The largest shadow price occurs for the element that maximizes \( |g_i|^2 / p_i \). Thus, a low-power element which experiences large gain to the beamformed receiver would do well to increase its power beyond its constraint, if possible.

C. Gauss-Seidel MMSE Algorithm

Recall that the cyclic algorithm attempts to find a jointly optimal pair \( (z^*, w^*) \) by sequentially optimizing \( z \) and \( w \). Optimization of \( w \) can be carried out using (10) and \( z \) can be optimized using Proposition 1. However the situation may arise where \( ||H^Hw||_p > 1 \). But from Proposition 4 we know that in such a case the MSE can be reduced by updating \( w \) to \( \hat{w} = w / ||H^Hw||_p \), and then updating \( z \) to \([H^H\hat{w}]^p \). But since \([H^H\hat{w}]^p = [H^Hw]^p \), there is no need to update \( w \) to \( \hat{w} \). Thus the cyclic algorithm only needs to sequentially update \( z \) and \( w \) using \( z = [H^Hw]^p \) and (10). The transmit vector can be initialized to \( z^{(0)} = [\xi]^p \) (see Section IV-A).

V. MULTICARRIER PAPC BEAMFORMING WITH SUM-MSE OBJECTIVE

In the multicarrier case, the goal is to solve problem (6). Once again, joint optimization of transmit and receive coefficients is a non-convex problem and we appeal to the Gauss-Seidel method. Optimization of the receive weights \( w_1, \ldots, w_K \) given a set of transmit weights \( z_1, \ldots, z_K \) is a separable problem. Each carrier’s receive weight can be computed using (10). Thus, this portion of the Gauss-Seidel algorithm is easy. It remains to find the optimum PAPC-trans-
mit weights given a set of receive weights. The multicarrier precoder optimization problem is

\[
\text{minimize } \sum_{k=1}^{K} (z_k^H g_k z_k - 2\text{Re}(z_k^H g_k)) \\
\text{subject to } \sum_{k=1}^{K} |z_{i,k}|^2 \leq p_i, \; i = 1, \ldots, n. \tag{15}
\]

This problem is a convex QCQP and admits strong duality, as in the single carrier case, but the constraints introduce coupling in the variables. A primal gradient projection algorithm could be used, but the projection step would be difficult. On the other hand, the dual problem has feasible region \(\mathbb{R}_+^n\), for which projection is simple. Thus we solve the dual problem with a gradient projection algorithm. The dual function, derived in Appendix B, is

\[
d_\lambda = \begin{cases} 
- \sum_k g_k^H A^{-1} g_k - \lambda^T p, & \lambda \in \mathbb{R}_+^n \\
- \sum_k (\frac{1}{1 + g_k^H A^{-1} g_k} - \lambda^T p), & \lambda \in \partial \mathbb{R}_+^n
\end{cases}
\]

where \(A = \text{diag}(\lambda_1, \ldots, \lambda_n)\). The dual problem is

\[
\text{maximize } d_\lambda \quad \lambda \in \mathbb{R}_+^n
\]

The dual function \(d_\lambda\) is concave and differentiable on \(\mathbb{R}_+^n\) and projection onto the orthant is simple. Primal variables are calculated with the Lagrangian minimizers given in (21) and (22).

We now have the necessary tools to jointly optimize \(\{z_k, w_k\}\). In Algorithm 1 optimization steps for \(\{z_k\}\) and \(\{w_k\}\) are alternated. First the transmit weights are initialized to the projected dominant eigenvectors of each carrier and then assigned equal power (Step 1). Then the cyclic iterations begin. Optimization of the receive weights \(\{w_k\}\) is a separable problem, where each carrier’s weight \(w_k\) is found using

\[
f(z, H, R_n) = R_n^H \mathbf{Hz}(1 + z^H H^H R_n^{-1} H^H)z^{-1}
\]

as in (10). Next, the dual variables \(\lambda\) are initialized and updated using gradient ascent on the dual function followed by projection on the orthant \(\mathbb{R}_+^n\). The step size \(\alpha_j\) is found using a line search. Finally, the updated transmit weights are found from the Lagrangian minimization equation (21) as shown in Step 9, or (22) if \(\lambda\) lies on the boundary of the orthant. Note \(\Lambda_j = \text{diag}(\lambda_j)\).

### VI. Numerical Example

In this section we demonstrate the utility of the cyclic algorithm through a numerical example. We consider two distributed groups of nodes with \(n = 20\) transmitters and \(m = 10\) receivers. The transmit power constraints (in Watts) are chosen uniformly in [0.1, 1.0] and the receive noise variances (in dBW) are uniform with spread 10 dB. The noise variance is frequency independent so \(R_{n,k} = R_n\) and the noise between receivers is uncorrelated so \(R_{k,k}\) is diagonal. The system uses \(K = 128\) carriers over \(W = 10\) MHz centered at \(f_c = 2\) GHz. We assume the receivers are spaced sufficiently far apart such that for each transmitter/receiver pair, we have an independent Rayleigh channel with exponential intensity profile and delay spread of 4\(\mu s\).

We compare the multicarrier cyclic algorithm to three suboptimal approaches and to the optimum total-power-constrained weights \(z_k^{(P_T)}\) [16, eqn 12]. The latter approach provides a lower bound on MSE since its feasible region contains the feasible region of (15). The first suboptimal approach uses \(z_k = K^{-1/2}[c_k]^T\). That is, for each carrier \(k\) we find the optimum unconstrained (without PAPC) weight, which is the dominant eigenvector of \(H_k\), and project this weight on the feasible set. No attempt is made to optimally allocate power across carriers, and instead all carriers are allocated equal power. Note that this choice of transmit vectors is also used to initialize the cyclic algorithm (Step 1 of Algorithm 1). In the second suboptimal method, the single-carrier cyclic algorithm from Section IV-C is run independently for each carrier \(k\) and once again equal power is assigned to each carrier. Thus this approach should yield better performance than the first, since each carrier’s transmit and receive weights are jointly optimized. But once again the transmit power is not allocated optimally across carriers. Thirdly we consider a naive approach where we find \(z_k^{(P_T)}\) and for every antenna in violation of its per-antenna constraint, scale the weight magnitude (equally across carriers) so that the constraint is met with equality. In implementing Algorithm 1, we terminate the cyclic loop after 20 iterations and the dual gradient loop after 200 iterations.

In Figure 1 we plot the empirical cumulative distribution functions (CDFs) of the carrier SNR, with 400 Monte-Carlo trials, using the approaches described above. The figure shows that the single-carrier cyclic algorithm is only marginally better than the projected eigenvector approach. This is consistent with results reported for the max-gain case which suggest that the projected dominant eigenvector is nearly optimal for the single-carrier problem [12]. Returning to the figure, we see that the cyclic multicarrier algorithm improves median SNR by approximately 0.7 dB relative to the first two suboptimal approaches. The naive method of scaling the out of tolerance weights of \(z_k^{(P_T)}\) performs very poorly.

The total-power-constrained approach performs 0.2 dB better than the multicarrier cyclic algorithm, but this marginal improvement comes at a steep price in terms of per-antenna
QCQP and another Gauss-Seidel algorithm was developed for weights for multicarrier systems with sum-MSE objective is not encumbered by this norm condition. Optimum transmit also developed a Gauss-Seidel algorithm for joint MMSE certain norm condition on the effective MISO channel. We showed that the optimum transmit weights under PAPC are nearly identical to the gain maximizing weights under a constraint violations. Figure 2 sheds light on these violations by showing the CDFs for the number of antennas in violation and for the maximum percent violation. There are always at least 7 antennas in violation and the median value of the maximum percent violation is well over 200%. Thus this method is not feasible with per-antenna constraints.

VII. CONCLUSION

We have presented a method for joint transmit and receive beamforming optimization with MSE objective and nonuniform PAPC for both single and multicarrier systems, which have applicability for distributed beamforming systems. We also developed a Gauss-Seidel algorithm for joint MMSE optimization of the transmitter and receiver weights, which is not encumbered by this norm condition. Optimum transmit weights for multicarrier systems with sum-MSE objective were found using a dual gradient algorithm which solves a QCQP and another Gauss-Seidel algorithm was developed for the multicarrier case. Finally, we showed through numerical example the benefits of the method by comparison to several suboptimal approaches. The numerical example also demonstrated the disadvantage of a total-power-constrained approach.

APPENDIX A

PROOFS OF PROPOSITIONS 1–3

Proof of Proposition 1

First we note that the problem (13) can be converted into an equivalent problem with all real variables as in [19]. Define the ~ operator which maps complex vectors in C^n to real vectors in R^{2n}, and complex matrices in C^{n×n} to real matrices in R^{2n×2n} according to:

\[ \tilde{x} = \begin{bmatrix} \text{Re}(x) \\ \text{Im}(x) \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \text{Re}(A) & -\text{Im}(A) \\ \text{Im}(A) & \text{Re}(A) \end{bmatrix}. \]

It is easy to show that \( \tilde{A} \) is positive semidefinite if and only if \( A \) is positive semidefinite. Now (13) can be converted to the following real problem:

\[
\begin{align*}
\min_{z \in \mathbb{R}^{2n}} & \quad z^T \hat{G} z - 2 z^T \hat{g} \\
\text{subject to} & \quad \tilde{z}^T \hat{E}_i \tilde{z} \leq p_i, \quad i = 1, \ldots, n.
\end{align*}
\]

Since \( \tilde{G} \) and \( \hat{E}_i \) are positive semidefinite, \( \hat{G} \) and \( \hat{E}_i \) are also positive semidefinite. Problem (16) has a convex objective and \( n \) convex inequality constraints. Therefore it is a convex problem [13]. The Lagrangian function is given by

\[
\mathcal{L}(\tilde{z}, \lambda) = \tilde{z}^T \hat{G} \tilde{z} - 2 \tilde{z}^T \hat{g} + \sum_{i=1}^{n} \lambda_i (\tilde{z}^T \hat{E}_i \tilde{z} - p_i)
\]

where \( \mathbf{p} = [p_1, \ldots, p_n]^T \) and \( \lambda = [\lambda_1, \ldots, \lambda_n]^T \) are the Lagrange multipliers associated with the magnitude constraints. Finally \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \) and \( \hat{\Lambda} = \text{blkdiag}(\Lambda, \Lambda) \).

Since (16) is a convex problem and a strictly feasible \( \tilde{z} \) exists (for example, the vector \([\epsilon, 0, \ldots, 0]^T\) with \( \epsilon < \sqrt{\frac{1}{p_1}} \)), Slater’s condition is satisfied, the problem admits strong duality, and the following Karush-Kuhn-Tucker (KKT) conditions are thus sufficient for the primal-dual pair \((\tilde{z}^*, \lambda^*)\) to be optimal [13]:

1) \( \nabla_{\tilde{z}} \mathcal{L}(\tilde{z}^*, \lambda^*) = 0 \)
2) \( \tilde{z}^* \) satisfies all inequality constraints in (16)
3) \( (\tilde{z}_{i+1}^* + \tilde{z}_{i+2}^* - p_i)\lambda_i^* = 0, \quad i = 1, \ldots, n \)
4) \( \lambda^* \succeq 0 \)

From (17) the zero-gradient condition is \((\hat{G} + \hat{\Lambda}) \tilde{z} = \hat{g}\). Thus, in terms of complex variables, KKT condition 1 can be written \((\hat{G} + \hat{\Lambda}) z = g\), or

\[
(\mathbf{gg}^H + \hat{\Lambda}) z = g
\]

which is equivalent to

\[
\lambda_i z_i = g_i (1 - \sum_k g_k^* z_k), \quad i = 1, \ldots, n.
\]

It is easily verified that this equation is satisfied for

\[
z_i = \sqrt{p_i} e^{j\varphi_i}, \quad \lambda_i = \frac{|g_i|}{\sqrt{p_i}} (1 - ||g||_p).
\]
Furthermore, this choice of $z_i$ has magnitude $\sqrt{p_i}$. Thus the first three KKT conditions hold. By the assumption $\|g\|_p \leq 1$, the Lagrange multipliers are non-negative. Thus the solution (19) is optimal. \hfill $\square$

**Proof of Proposition 2**

Letting $\lambda_i = 0$ for all $i$, KKT conditions 3 and 4 are immediately satisfied. Letting $z_i = \sqrt{p_i} (\sum_k \sqrt{p_k})^{-1} (g_i^*)^{-1}$, (18) is again easily verified, satisfying condition 1. Finally assuming $\min g_i \geq (\sum_k \sqrt{p_k})^{-1}$, we get $\max \lambda_i |z_i| \leq \sqrt{p_i}$, which satisfies condition 2. This proves Proposition 2. \hfill $\square$

**Proof of Proposition 3**

Apply KKT conditions again.

**APPENDIX B**

**DERIVATION OF MULTICARRIER DUAL FUNCTION**

The primal problem given in (15) can be converted to a real problem using the method in Appendix A. This problem has Lagrangian

$$\mathcal{L}(\tilde{z}_1, \ldots, \tilde{z}_K, \lambda) = \sum_{k=1}^{K} (\tilde{z}_k^T (\tilde{G}_k + \tilde{\Lambda}) \tilde{z}_k - 2 \tilde{z}_k^T \tilde{g}_k) - \lambda^T p$$

which is quadratic in $\tilde{z}$. The dual function is the infimum of the Lagrangian over all transmit vectors. The matrix coefficient $\tilde{G}_k + \tilde{\Lambda}$ is positive definite when all Lagrange multipliers are positive and positive semidefinite when at least one Lagrange multiplier is zero. (We assume dual feasibility so no Lagrange multipliers are negative). The Lagrangian is minimized when $(\tilde{G}_k + \tilde{\Lambda}) \tilde{z}_k = \tilde{g}_k$ for all $k$, or, in terms of the complex variables, when

$$((g_k g_k^H + \Lambda) z_k = g_k, \quad k = 1, \ldots, K). \quad (20)$$

In the positive definite case ($\lambda \in \mathbb{R}^n_+$) the minimizers are

$$z_k^* = (g_k g_k^H + \Lambda)^{-1} g_k$$

$$= \frac{\Lambda^{-1} g_k}{1 + g_k^H \Lambda^{-1} g_k}. \quad (21)$$

Inserting these values into the Lagrangian gives the dual function for $\lambda \in \mathbb{R}^n_+$

$$d(\lambda) = -\sum_k \frac{g_k^H \Lambda^{-1} g_k}{1 + g_k^H \Lambda^{-1} g_k} - \lambda^T p.$$ (22)

For the positive semidefinite case, assume there is a zero-valued Lagrange multiplier: $\lambda_d = 0$. Then either of the following sets of vectors satisfy (20) and thus minimize the Lagrangian

$$z_k^* = \frac{1}{g_{k,q}} e_q, \quad z_k^* = \frac{g_{k,q} ||g_{k,q}||^2}{|g_{k,q}|^2} e_q, \quad k = 1, \ldots, K$$

(where $g_{k,q}$ is the $q$th component of $g_k$ and $e_q$ is the $q$th standard basis vector of $\mathbb{R}^n$). The dual function is thus

$$d(\lambda) = -K - \lambda^T p$$

when $\lambda$ is on the boundary of the non-negative orthant $\partial \mathbb{R}^n_+$. The dual function is differentiable on $\mathbb{R}^n_+$ since the minimizers in (21) are unique [20]. However, we cannot claim differentiability when $\lambda$ is on the boundary of the orthant since the minimizers in (22) are not unique.

**REFERENCES**


