Compressive Wideband Spectrum Sensing based on Cyclostationary Detection

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Abstract—Cognitive Radios (CRs) operating in a wideband channel must detect the presence of licensed users within a constrained sensing time, and under energy limitations. In this paper, we explore a reduced-complexity compressive sampling cyclostationary detection method which exploits the sparsity of the two-dimensional spectral correlation function (SCF). We obtain a closed-form expression that reconstructs the Nyquist SCF from the sub-Nyquist samples upon which the signal detection is performed. Bounds on the minimum sampling rate resulting in a unique reconstruction for a given SCF sparsity are quantified theoretically. Due to the additional sparsity introduced in the SCF with respect to the power spectral density, the minimum sampling rates for cyclostationary detection are shown to be lower than those required for energy detection.

I. INTRODUCTION

Motivated by the spectrum scarcity problem, Cognitive Radios (CRs) have been proposed as a way to opportunistically communicate over unused spectrum licensed to Primary Users (PUs). In this context, secondary users (SUs) sense the spectrum to detect the presence or absence of PUs, and use the unoccupied bands while maintaining a predefined probability of mis-detection. Conventional signal processing approaches for spectrum sensing assume detection of a narrowband signal in noise. Most commonly used techniques include matched filtering, energy detection, and cyclostationary detection [1]–[4]. In this paper we consider wideband spectrum sensing using cyclostationary feature detection since matched filtering yields a high complexity architecture, and energy detection is sensitive to noise and fails to differentiate among signals. Cyclostationary or feature detection [5], relies on the detection of embedded cyclostationary features of modulated signals. This approach is robust against noise uncertainty since it exploits the stationary property of the noise process and hence asymptotically separates signal from noise, whereas energy detection fails to differentiate signal from noise in low SNR regimes. In addition, feature detectors are able to differentiate among signals with different modulation types.

Wideband spectrum sensing is highly desirable since sensing multiple channels simultaneously increases the probability of finding unused spectrum. Implementation of wideband spectrum sensing radio can be realized using: 1) one narrowband receiver with tunable RF frequency (only one band can be sensed at the time), 2) parallel narrowband receivers operating in fixed frequency bands (entire wideband spectrum can be sensed), 3) a wideband receiver. A wideband receiver allows spectrum channelization in digital domain, which is flexible and less expensive. Unfortunately, the radios that operate over a channel bandwidth on the order of 500 MHz or more require high sampling rate A/D converters with large dynamic range due to in-band PU signals. However, high sampling rate A/D converters are hard to design and consume high power.

In order to reduce sampling rate requirements, it is appealing to use a compressive sensing approach that samples wideband signals below the Nyquist rate, given that the received wideband signal is sparse in a given domain. Recently proposed compressive sensing analog front-end architectures include Analog to Information Converters (AIC) [6], Modulated Wideband Converters (MWC) [7]–[9], and other parallel mixed-signal architectures [10]–[12]. Compressive sampling requires signal reconstruction that is realized in DSP, and often involves solving complex optimizations such as Orthogonal Matching Pursuit (OMP) and the ESPRIT algorithm [13]. Further, reconstruction methods such as the ones in [14], [15] rely on numerical methods to solve an optimization problem, which make them difficult to implement in VLSI.

In this paper, we consider a cyclostationary detection approach for wideband spectrum sensing that exploits sparsity in the cyclic and angular frequency domains. Compressive sampling is particularly appealing in this case because spectral correlation features of modulated signals result in a few distinct peaks in the cyclic frequency domain while noise does not exhibit any spectral correlation peaks asymptotically. In order to detect spectral correlation peaks of a PU signal, feature detectors first estimate the Spectral Correlation Function (SCF) [16]. Conventional detectors include single-cycle and multicycle detectors which collect the energy in the SCF at one or many cyclic frequencies, respectively.

One approach to perform sub-Nyquist cyclostationary feature detection is to first recover the Nyquist samples, then estimate the SCF, and perform feature detection. Such an approach is presented in [17] using the MWC as a frontend. In the very first paper that considered cyclostationary detection directly from sub-Nyquist samples [18], the SCF reconstruction is performed blindly with no a priori knowledge of the carriers and bandwidths of the signals to be detected, and detection is performed on a per-band basis using a GLRT statistic. In this paper we exploit the fact that in a typical CR setting, the sensing radios have some information about the signals to be detected. In other words, the goal of the CRs is to detect the presence or absence of one or many PUs simultaneously in a wideband channel, with a priori knowledge of their carrier frequencies, symbol rates, and modulation schemes. Similar to our previous work in [19], we explore a reduced complexity approach for cyclostationary detection that exploits the knowledge about carrier frequencies.
and signal bandwidths of the signals to be detected, and
reconstructs the spectral correlation peaks of the SCF without
reconstructing the Nyquist samples. The optimization exploits
the SCF sparsity and yields a closed-form solution to the
reconstruction of the SCF-based test statistic. In addition,
to the best of the authors’ knowledge, there has been no
theoretical work with respect to the minimum achievable
compression ratios under a given SCF sparsity that guarantee
a unique solution to the reconstruction problem. This presents
a design guideline in terms of the achievable sampling rate
reductions.

The contributions of this work are twofold:

- We develop a closed-form low-complexity SCF recon-
  struction from sub-Nyquist samples which is applicable
to any analog-front end compressive sensing modulator
  which uses the same system model. The given algorithm
  reconstructs only useful spectral correlation peaks used
  for detection, making it energy efficient and imple-
  mentable in VLSI.

- We quantify the bounds on the minimum achievable
  compression ratio for a given spectral sparsity for both
cyclostationary and energy detection, which guarantee
  the uniqueness of the reconstruction.

We show that the proposed detector requires additional sens-
\( s = \frac{1}{N} \) ing time in order to improve detection performance up to
a threshold compression point below which the gains are
negligible with increasing sensing time. Under a noiseless
scenario and with enough samples to ensure the sparsity of the
reconstructed SCF, we show that sub-Nyquist sampling incurs
no loss in reconstruction with respect to Nyquist sampling
up to a compression ratio that we call compression wall and
theoretically quantify it.

The rest of the paper is organized as follows. In Section
II, we present our system model, give a brief overview on
cyclostationary-based detection and the problem statement.
In Section III, we relate the SCF of the sub-Nyquist samples
to the one using Nyquist samples, and formulate our
reduced complexity optimization problem for reconstruction
of the spectral peaks of interest. In Section IV we derive
the conditions necessary to guarantee a unique solution to the
optimization problem. Section VI gives the numerical results
with a detailed discussion on the same. Finally, Section VII
concludes the paper.

## II. System Model and Problem Formulation

### A. System Model

In the context of CR spectrum sensing, we focus on the
processing of a received wideband signal in a channel of
bandwidth \( B \) centered at some carrier frequency. The approach
and analysis considered here apply to any channel bandwidth,
not necessarily supported by state-of-the-art A/D converters.
Given the time and energy constraints on the sensing stage,
we consider a limited time window of length \( T_{\text{sensor}} = N_T T_s \)
during which the sensing radio acquires a total number of
\( N_T \) incoming samples, where \( T_s \) is the sampling period. Let

\[ t \in [0, T_{\text{sensor}}] \]
denote the time variable, and assume that
the wideband channel could be occupied by \( K \) PU signals
\[ s_k(t) \] \( \forall k \in [1, \ldots, K] \). Under hypothesis \( H_{k,0} \), \( H_{k,1} \), the \( k^{th} \)
PU is defined as being absent or active, respectively. Therefore,
the received wideband signal is given by

\[ x(t) = \sum_{k=1}^{K} s_k(t) \], \quad \text{where } 0 \leq t \leq T_{\text{sensor}} \text{ and} \]

\[ s_k(t) = \begin{cases} \frac{1}{2} \Re \{ a_k(nT_k)p_k(t-nT_k)e^{j2\pi f_{c_k}t} \} + w_k(t), & \text{under } H_{k,0} \\ \frac{1}{2} \Re \{ a_k(nT_k)p_k(t-nT_k)e^{j2\pi f_{c_k}t} \} & \text{under } H_{k,1} \end{cases} \]

(1)

where \( a_k(nT) \) and \( p_k(t) \) are the transmitted information
symbols and the pulse shaping filter of the \( k^{th} \) transmitted signal
respectively, and \( w_k(t) \) is the AWGN in the band occupied
by the \( k^{th} \) transmitter. We assume transmitted information
symbols with average power \( \sigma_{a_k}^2 \), a pulse shape filter \( p_k(t) \) of
unit energy, and we define the Signal to Noise Ratio by

\[ \text{SNR} = \frac{\sigma_{a_k}^2}{\sigma_{w_k}^2} \]

where \( \sigma_{w_k}^2 \) is the noise variance in the channel
occupied by \( s_k(t) \).

In the remaining of this paper, we assume perfect knowledge
of the carrier frequencies \( f_{c_k} \) and symbol rates \( 1/T_k \)
of the PUs that might be active in the wideband channel. In addition,
we assume the knowledge of the modulation scheme being
used by each of the PUs. During the sensing operation, the
CR detects which of the \( K \) PUs are active.

### B. Cyclostationary Detection based on Nyquist Sampling

Before processing, the wideband channel is downconverted
to baseband and sampled. Given that the computation of the
SCF includes computing the auto-correlation averaged over
different frames of samples, we consider a frame-based model,
where each frame is of length \( N_s \) samples, and where the
remaining \( N_T/N \) frames (assumed to be integer) are used
for statistical averaging. Let \( x \in \mathbb{R}^{N_s} \) denote a single frame of
samples obtained from sampling \( x(t) \) at the Nyquist rate of
\( f_s = B \).

In the context of time constrained spectrum sensing, due to
the limited number of samples acquired, perfect reconstruction
of the SCF is not possible. Thus, the spectrum sensing process
estimates the non-asymptotic SCF based on the \( N_T \) samples
acquired as follows

\[ \tilde{S}_x = F_N \left( \frac{N_f}{N_T} \sum_{\ell=1}^{N_f} X_{\ell} X_{\ell}^T \right) F_N, \]

(2)

where \( N_T/N \) is the number of spectral averages, \( X_{\ell} \) is the
\( \ell^{th} \) data frame received, and \( F_N \) is a \( N \times N \) DFT matrix.

The frame length \( N \) determines the SCF resolution in both

| Table I: Cyclic Features for Some Modulation Classes |
|-----------------|-----------------|-----------------|
| Modulation      | Peaks at \( (\alpha, \jmath) \) |
|-----------------|-----------------|-----------------|
| BPSK            | \((1, f_r)\), \((2f_c, 0)\), \((2f_c \pm \frac{f_r}{2}, 0)\) |
| MSK             | \((1, f_r)\), \((2f_c, 0)\), \((2f_c \pm \frac{f_r}{2}, 0)\) |
| QAM             | \((1, f_r)\) |

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  the uniqueness of the reconstruction.
angular and cyclic frequency (being 1/NTs), and the number of frames NTs/N emulates the expectation over different realizations. Note that as NTs/N \to \infty, the noise fluctuations are smoothed out and the estimated discrete SCF approaches the asymptotic continuous SCF [5], which is a sparse two-dimensional spectral map with non-zero features at the cyclic / angular frequency pairs shown in Table I for different modulation classes [5] and in the PSD corresponding to \alpha = 0. Note that AWGN is a widesense stationary process and exhibits no cyclic correlation. Therefore, the SCF of noise asymptotically has no spectral features at \alpha \neq 0.

Conventionally, Nyquist-based detectors for which Ts = 1/B consider the problem of detecting the presence/absence of one or multiple (up to K) signals in the band of interest. Here, we use a single-cycle detector which collects the energy of the SCF at a single cyclic frequency \alpha. The noise in the wideband channel is assumed to be AWGN. Also, since cyclostationary detection is robust to interference from strong adjacent band signals, the received power from all the transmitters is assumed to be equal, yielding the same in-band SNR for all signals. Therefore, the detection performance across the bands of interest will be the same for all K signals. For this reason, we drop the index k in the two hypotheses, and let \mathcal{H}_0 and \mathcal{H}_1 correspond to the hypotheses that the desired signal sk(t) being absent or present respectively for all indices k \in [1, ..., K]. The detection is performed as a standard hypothesis test on the test statistic based on the energy in SCF for a given cyclic frequency \alpha and angular frequency f. The test statistic \mathcal{T} = |\mathbf{S}_x(\alpha, f)|^2 obtained from (2) at a given (\alpha, f) pair is compared against a threshold \gamma. The objective of the detector is to meet a desired probability of false alarm and probability of detection point on the receiver operating curve (ROC) curve, defined as Pfa = \Pr(\mathcal{T} > \gamma|\mathcal{H}_0), Pd = \Pr(\mathcal{T} > \gamma|\mathcal{H}_1) respectively.

C. Acquiring Sub-Nyquist samples

Motivated by the two-dimensional sparsity in the SCF domain, we proposed to sample below the Nyquist rate and perform spectrum sensing using sub-Nyquist samples. Reducing the sampling rates would make the A/D converters easier to design and reduce their energy consumption. For a fixed T_{sense}, instead of acquiring NT samples at a high rate fs, we acquire M_T samples at a rate \frac{M_T}{NTs} \leq f_s. We again perform frame-based processing, where we let \mathbf{z} \in \mathbb{R}^M denote a frame of sub-Nyquist samples. The M_T samples will result in M_T/M different frames to be processed for detection. We further assume that a single sub-Nyquist frame \mathbf{z} \in \mathbb{R}^M is related to a Nyquist frame x \in \mathbb{R}^N via \mathbf{z} = \mathbf{A}x, where \mathbf{A} \in \mathbb{R}^{M \times N} is the sampling matrix. As a result of the matrix multiplication, the compression ratio is defined as \frac{M_T}{NTs}. Note that for practical implementation of the compressive sensing algorithms, the vector \mathbf{z} can be obtained directly from any of the analog front-ends [6], [11], [12]. In this work, \mathbf{A} is chosen to be a random Gaussian matrix in \mathbb{R}^{M \times N} with i.i.d. components. Using the sub-Nyquist samples, the Nyquist SCF is reconstructed by solving a standard least-squares problem, using which the signal detection is performed.

The aim of this work is to 1) understand the tradeoffs between compression, sensing time, and the resulting sensing performance, 2) reduce the complexity of the SCF reconstruction by making use of the known signal parameters, namely the modulation class, the carrier frequency, and the symbol rate of the PU that we wish to detect the presence of, and express the solution of the optimization problem in closed-form, and 3) derive the conditions needed to guarantee uniqueness of the reconstructed SCF along with probabilistic measures to obtain the sparsest unique solution.

III. NYQUIST-SCF RECONSTRUCTION USING SUB-NYQUIST SAMPLES

In this section, we propose a wideband signal detection algorithm that reconstructs specific points of the SCF from samples obtained via a sub-Nyquist compressed sampling receiver and detects the presence or absence of sk(t) for any given index k. We derive an expression relating the Nyquist and sub-Nyquist SCF which is subsequently used to formulate an optimization problem for reconstruction. In fact, given that we know the modulation scheme, carrier frequency and symbol rate of the transmitted signal sk(t), reconstructing the entire SCF from the sub-Nyquist samples is not necessary, reducing the optimization to a standard least-squares problem which can be solved in closed-form.

A. Relating Nyquist and Sub-Nyquist SCF

Since the SCF requires finding the autocorrelation of the signal being processed, we start by defining the non-asymptotic auto-correlation matrices \mathbf{R}_x \in \mathbb{R}^{N \times N} and \mathbf{R}_z \in \mathbb{R}^{M \times M}

\begin{equation}
\mathbf{R}_x = \frac{1}{L} \sum_{\ell=1}^{L} \mathbf{x}_\ell \mathbf{x}_\ell^H \quad \text{and} \quad \mathbf{R}_z = \frac{1}{L} \sum_{\ell=1}^{L} \mathbf{z}_\ell \mathbf{z}_\ell^H,
\end{equation}

where L = M_T/M frames of \mathbf{z} and L = NTs/N frames of x are considered for statistical averaging of the covariance matrices, \mathbf{x}_\ell \in \mathbb{R}^N is a frame of Nyquist samples and \mathbf{z}_\ell \in \mathbb{R}^M is a frame of sub-Nyquist samples. Given that our compressive sampling model is defined as \mathbf{z} = \mathbf{A}x, it follows that \mathbf{R}_x and \mathbf{R}_z are related via

\begin{equation}
\mathbf{R}_z = \mathbf{A} \mathbf{R}_x \mathbf{A}^H
\end{equation}

Vectorizing both sides of the equation gives the following

\begin{equation}
\mathbf{r}_z = (\mathbf{A} \otimes \mathbf{A})^H \mathbf{r}_x \triangleq \Phi \mathbf{r}_x
\end{equation}

where \otimes is the Kronecker product, \mathbf{r}_x = \text{vec}(\mathbf{R}_x), \mathbf{r}_z = \text{vec}(\mathbf{R}_z) and \Phi = \mathbf{A} \otimes \mathbf{A} \in \mathbb{C}^{M^2 \times N^2}

The SCF matrix \mathbf{S}_x can be obtained from the auto-correlation matrix \mathbf{R}_x by taking Fourier transform of \mathbf{R}_x with respect to its columns on the right and rows on the left. This can be expressed as

\begin{equation}
\mathbf{S}_x = \mathbf{F}_N \mathbf{R}_x \mathbf{F}_N^H
\end{equation}
where $F_N$ is a $N \times N$ DFT matrix. Again performing the vec operation on this equation gives

$$s_x = T_x r_x \quad (7)$$

where $s_x = \text{vec}(S_x)$ and $T_x = F_N \otimes F_N \in \mathbb{C}^{N^2 \times N^2}$. The matrix $T_x$ consists of two FFT matrices $F_N$ which are responsible for the transformation of the rank since they are composed of DFT matrices which are also full-rank matrix pairs of $r_x$ to the $(\alpha, f)$ pair in the SCF. Similarly, we obtain $s_x = T_x r_x$ where $s_x = \text{vec}(S_x)$, $T_x = F_M \otimes F_M \in \mathbb{C}^{M^2 \times M^2}$ and $F_M$ is a $M \times M$ DFT matrix.

The $T_x$ and $T_x$ matrices are full rank non-singular matrices since they are composed of DFT matrices which are also full-rank non-singular and $rank(A \otimes B) = rank(AB)$. Hence, $T_x^H T_x = T_x T_x^H = I_{N^2 \times N^2}$ where $\dagger$ denotes the pseudo-inverse operation. Also, $T_x$ and $T_x$ are orthogonal matrices since they are formed using DFT matrices. This can be seen as follows

$$(F_N \otimes F_N)^T (F_N \otimes F_N) = (F_N^T \otimes F_N^T) (F_N \otimes F_N) = (F_N^T \otimes F_N^T) (F_N \otimes F_N) = I_N \otimes I_N \otimes Z_N^2 \otimes Z_N^2$$

Now, as $T_x$ is an orthogonal matrix, $T_x^H = T_x^T$ and hence $T_x^H T_x = T_x T_x^H = I_{N^2 \times N^2}$. We can now relate $s_x$ and $s_x$ using equation (5) as follows

$$T_x r_x = T_x \Phi r_x \quad (8)$$

where $\Phi = T_x \Phi T_x^H \in \mathbb{C}^{M^2 \times N^2}$. Eq. (8) relates the vectorized SCF of the sub-Nyquist samples to that of the Nyquist samples, which we want to reconstruct. In order to ensure stable SCF reconstruction, the measurement matrix $\Phi$ must satisfy the restricted isometry property (RIP) as has been shown by [20], [21]. In the next section, we show that the way we construct $\Phi$ in our algorithm guarantees that it will satisfies the RIP with high probability.

**B. Restricted Isometry Property of the Measurement Matrix $\Phi$**

Prior works have verified that RIP holds with high probability for matrices whose entries are independent and identically distributed (i.i.d.) realizations of certain random distributions like the Gaussian distribution [22]. In our algorithm, the measurement matrix is $\Phi$ which can be expressed as follows

$$\Phi = T_x \Phi T_x^H = (F_M \otimes F_M) (A \otimes A) (F_N \otimes F_N)^H = (F_M \otimes F_M) (A \otimes A) (F_N^H \otimes F_N^H) = (F_M A F_N^H) \otimes (F_M A F_N^H) \quad (9)$$

where the last two equalities come from the properties of Kronecker product of matrices. Hence, one of the terms in $\Phi$ is $(A \otimes A)$, a Kronecker product of two i.i.d Gaussian matrices which we study first. From [23] we have

$$\delta_K(\Psi_1 \otimes \Psi_2 \otimes \ldots \otimes \Psi_D) \geq \max_{1 \leq d \leq D} \delta_K(\Psi_d)$$

where $\delta_K(\Psi_1), \ldots, \delta_K(\Psi_D)$ are the restricted isometry constants of the matrices $\Psi_1, \ldots, \Psi_D$. Hence, $\delta_K(A \otimes A) \geq \delta_K(A)$. Further, [24] gives an upper bound on the RIP constant of Kronecker product of matrices

$$\delta_K(\Psi_1 \otimes \Psi_2 \otimes \ldots \otimes \Psi_D) \leq \prod_{d=1}^{D} (1 + \delta_K(\Psi_d)) - 1$$

Hence, $\delta_K(A \otimes A) \leq \delta_K(A)^2 + 2 \delta_K(A)$. As $A$ is an i.i.d Gaussian matrix satisfying RIP with high probability with $0 < \delta_K(A) < 1$, thus $\delta_K(A)^2$ is very small and $\delta_K(A)^2 + 2 \delta_K(A) \approx 2 \delta_K(A)$. Therefore, $\delta_K(A \otimes A)$ is bounded as follows

$$\delta_K(A) \leq \delta_K(A \otimes A) \leq 2 \delta_K(A) \quad (10)$$

The resulting pair of bounds is tight which implies that $A \otimes A$ also satisfies RIP with high probability. Now, from equation (9), $\Phi = (F_M A F_N^H) \otimes (F_M A F_N^H)$, the final step to prove that $\Phi$ satisfies RIP is to prove that the elements of $F_M A F_N^H$ are i.i.d with Gaussian distribution. This is shown using the following lemma.

**Lemma 1.** Let $A$ be a $M \times N$ full rank random matrix whose elements are all i.i.d with Gaussian distribution, and let $B \in \mathbb{C}^{N \times N}$, $C \in \mathbb{C}^{M \times M}$ be unitary matrices. Then the elements of $AB$ and $CA$ are also i.i.d Gaussian distributed with a different variance than the elements of $A$.

**Proof:** We know that given two random variables $x$ and $y$ with distributions $f_x(x)$ and $f_y(y)$ respectively, their joint distribution $f_{xy}(xy) = f_x(x) f_y(y)$ iff $x$ and $y$ are independent random variables.

Given a i.i.d Gaussian distributed matrix $A$, the joint element density is expressed as

$$f(a_{11}, a_{12}, \ldots, a_{mn}) = \frac{1}{(2\pi \sigma^2)^{mn}} \exp \left( -\frac{1}{2 \sigma^2} a_{11}^2 + a_{12}^2 + \ldots + a_{mn}^2 \right)$$

where $A_{i,j} \sim (0, \sigma^2) \forall i, j \text{ and } |||A||F$ is the Frobenius norm. Now, using the fact that the Frobenius norm is invariant under unitary transformation, i.e. $||AB||F = ||CA||F = ||A||F$, it follows directly from the joint element density of $A$ that the elements of $AB$ and $CA$ are also i.i.d Gaussian distributed.

From Lemma 1, it is clear that the elements of $F_M A F_N^H$ are i.i.d Gaussian distributed when $A$ is i.i.d Gaussian distributed since $F_M$ and $F_N$ are unitary DFT matrices. Hence, from (10), the Kronecker product of two i.i.d Gaussian matrices satisfies RIP, hence the measurement matrix $\Phi$ satisfies the RIP with high probability ensuring stable reconstruction of the Nyquist SCF $s_x$.

**C. Reconstruction of the Nyquist SCF**

The vectorized Nyquist SCF $s_x$ which is sparse in both $\alpha$ and $f$ can be reconstructed using (8). We formulate the
reconstruction as a standard regularized least squares problem by introducing an $\ell_1$ minimization term, namely
\[
\min_{\tilde{s}_x} ||s_x||_1 + \lambda ||s_x - \tilde{\Phi}s_x||_2^2. \tag{11}
\]
for some $\lambda > 0 \in \mathbb{R}$.

As was shown in Section II-B, the spectral correlation peaks are discrete in the cyclic frequency domain $\alpha$. The resolution in both $f$ and $\alpha$ is solely determined by the FFT size $N$, and is equal to $f_s/N$ where $f_s$ is the Nyquist sampling rate.

Therefore, in order to detect signals in a wideband channel, the resolution required for the spectral correlation peaks to be prominent would be high, leading to a large $N$. This leads to a large computational complexity in (11) since $s_x$ is in $\mathbb{C}^{N^2}$. Due to the sparsity of the vectorized SCF $s_x$, we need to reconstruct only the points of the SCF at which spectral correlation peaks would be present if the $k^{th}$ PU is active, hence making the reconstruction problem more computationally efficient.

Let $M_f$ be a matrix $\in \mathbb{R}^{K_f \times N^2}$ with elements equal to 1 at the indices of possible cyclic features. The reduced dimensionality SCF is defined as
\[
\hat{s}_x = M_f s_x, \tag{12}
\]
where $\hat{s}_x \in \mathbb{C}^{K_f \times 1}$. The number of features $K_f$ is a function of the number of signals present in the wideband channel, their bandwidths, the resolution of the SCF (determined by $N$ and the sampling rate) and the modulation class of the signals being detected (see Table I). Furthermore, we also define the matrix $\tilde{\Phi} = \tilde{\Phi}_{(j=1:K_f)}$, which selects the corresponding $K_f$ columns of $\hat{\Phi}$ and stores them in $\tilde{\Phi}$. Given that we are only reconstructing the non-zero points in the SCF vector, we can therefore drop the $\ell_1$ minimization term related to the SCF sparsity, and the unconstrained optimization problem from (11) becomes
\[
\min_{s_x} ||s_x - \tilde{\Phi}\hat{s}_x||_2^2, \tag{13}
\]
which is an optimization in $\mathbb{C}^{K_f}$, where $K_f \ll N^2$. This formulation renders the optimization problem more computationally efficient since the cardinality of the search space has been reduced from $N^2$ to $K_f$. The reconstruction in (13) is a regular least squares problem which can be solved in closed form as follows
\[
\hat{s}_x = \tilde{\Phi}^T \hat{s}_x \tag{14}
\]
It is well known that the solution to (14) is unique if the matrix $\tilde{\Phi}$ is full column rank. Since $\tilde{\Phi} = \tilde{\Phi}_{(j=1:K_f)}$, we analyze the measurement matrix $\tilde{\Phi}$ and derive the conditions which result in a unique solution to the proposed reconstruction problem, a core requirement of compressive sensing algorithms.

IV. UNIQUENESS OF THE COMPRESSIVE SENSING-BASED FEATURE RECONSTRUCTION

In this section, we study the conditions under which a unique solution exists for (14), as a function of the cardinality of the vector $s_x$, defined as
\[
C(s_x) = ||s_x||_0 = K_f. \tag{15}
\]
As the Nyquist SCF is sparse in both its angular and cyclic frequencies, $C(s_x)$ is therefore smaller than $N^2$. In fact, $C(s_x)$ is determined by the number of signals present in the wideband channel, and by their bandwidths. On the other hand, the location of the non-zero elements in $s_x$ is determined by the carrier frequencies and the bandwidths of the present signals.

Note that (15) assumes that there are $K_f$ non-zero elements in the Nyquist SCF when the number of averages $L$ in (3) are sufficient to make the SCF sparse. When more spectral correlation peaks are prominent due to limited number of samples, $K_f$ acts as a lower bound to the number of non-zero points in the SCF.

To study the uniqueness of the reconstructed vector, we start with the following two lemmas.

**Lemma 2.** Let $A$ be a $M \times N$ matrix formed from random i.i.d. components following a given distribution, then the rank of $A \otimes A = M^2$.

**Proof:** Given that the matrix $A$ is generated randomly, rank$(A) = M$ with high probability. The singular values of $A \otimes A$ are formed from all possible pairwise products of singular values of $A$, the total number of non-singular values for $A \otimes A$ is $M^2$, and therefore rank$(A \otimes A) = M^2$. \hfill $\square$

**Lemma 3.** Let $B$ be a $N^2 \times N^2$ full rank matrix, then the rank of $B^T$ is $N^2$.

**Proof:** From the Singular Value Decomposition (SVD), if $B = USV$, then $B^T = VS^T U^H = VS^T U$. Therefore, the number of non-singular values in $B^T$ are the same as in $B$, and therefore rank$(B^T) = $ rank$(B) = N^2$. \hfill $\square$

Lemma 2 can be used to obtain that rank$(T_x) = M^2$, and rank$(\tilde{\Phi}) = M^2$. Further, Lemma 3 can be used to prove that rank$(\tilde{T}_x^T) = N^2$. The matrix $\tilde{\Phi}$ is composed of $T_x \Phi T_x^T$. Hence, $\tilde{\Phi}$ is the product of three full-rank matrices and the rank$(\tilde{\Phi}) = \min(M^2, M^2, N^2) = M^2$ since $M \leq N$.

The above proof shows that there cannot exist a set of more than $M^2$ columns of $\tilde{\Phi}$ that are linearly independent. However, this does not mean that every set of $K_f$ columns of $\tilde{\Phi}$ are linearly independent, even if $K_f < M^2$. For this reason, we use the Spark of a matrix to find when a unique solution exists. The Spark of a matrix is defined as
\[
\text{Spark}(A) = \min_d ||d||_0 \text{ such that } Ad = 0. \tag{16}
\]
From the definition in (16), the Spark of a matrix is the smallest number of columns in any matrix such that the sub-matrix formed from the selected columns is not full-column rank. In other words, any subset of columns formed of less than the Spark of a matrix will be full column rank with probability 1. Obviously, the Spark of a matrix cannot be greater than its Rank + 1. The section below is aimed at finding the Spark of the matrix $\tilde{\Phi}$.

**A. Finding the Spark of Sensing Matrix**

In order to obtain the Spark of $\tilde{\Phi}$, we start by finding the matrices that compose it. From (9), we have that $\tilde{\Phi} = (F_M A F_N^H) \otimes (F_M A F_N^H)$. In order to obtain the Spark of $\tilde{\Phi}$, we start by finding the matrices that compose it. From (9), we have that $\tilde{\Phi} = (F_M A F_N^H) \otimes (F_M A F_N^H)$. 

Given that the sampling matrix $A$ is formed of i.i.d components, its Spark is equal to $M+1$ with high probability. In other words, any subset of $M$ columns of $A$ will form a full-column sub-matrix.

It was shown in [23] that if $A$ is a $M \times N$ rank-deficient matrix, i.e. $\text{Rank}(A) < N$, then the Spark$(A \otimes A) =$ Spark$(A)$. Therefore, in order to find Spark$(\tilde{\Phi})$, it is sufficient to find Spark$(F_M A F_N^H)$. From Lemma 1, we know that Gaussian distribution is invariant to unitary transformation. Hence, the elements of $(AF_N^H)$ are also i.i.d Gaussian distributed and thus Spark$(AF_N^H) =$ Spark$(A) = M + 1$.

**Lemma 4.** Let $A$ be a $N \times N$ invertible matrix, and let $B$ be a $N \times M$ rank deficient matrix. Then Spark$(AB) =$ Spark$(B)$.

**Proof:** Using the definition of Spark,

$$\text{Spark}(AB) = \min_d ||d||_0 \text{ such that } ABd = 0,$$

Given that matrix $A$ is invertible, the constraint therefore is equivalent to $Bd = 0$, and therefore,

$$\text{Spark}(AB) = \min_d ||d||_0 \text{ such that } Bd = 0,$$

which is the definition of Spark$(B)$. Therefore, Spark$(AB) =$ Spark$(B)$.

Let $T = AF_N^H$, where $T$ is a $N \times M$ rank deficient matrix. Given that $F_M$ is invertible since it is an $M$-point DFT matrix, Spark$(F_M T) =$ Spark$(T) = M + 1$ as a result of Lemma 4.

The above results prove that Spark$(F_M AF_N^H) =$ Spark$(A) = M + 1$. Hence, this proves that Spark$(\tilde{\Phi}) = M + 1$ since Spark$(F_M AF_N^H \otimes F_M AF_N^H) =$ Spark$(F_M AF_N^H)$ [23].

**B. Implications of the Spark(\tilde{\Phi}) and Uniqueness of the Solution**

As a result of the Spark of the matrix $\tilde{\Phi}$, we can guarantee that as long as $K_f < M + 1$, then the resulting matrix $\tilde{\Phi}$ will be full column rank. The uniqueness of the reconstruction with the same cardinality is guaranteed only when $K_f < (M+1)/2$. This is a direct result of the definition of Spark, i.e. if $x$ represents $z$, i.e $z = Ax$, it implies that all alternative representations of the same signal $z$ can be represented as $x + \delta$, for $\delta \in \text{Null}(A)$ where $\text{Null}(A)$ is the null-space of $A$. If we ensure that $||x||_0 < \text{Spark}(A)/2$, no vector $\delta$ (of the same cardinality as $x$) from the null-space of $A$ exists such that it can be added to $x$, nulling more entries in $x$. Hence, this representation is the sparsest one possible. The results about the Spark of $\tilde{\Phi}$ prove the following theorem.

**Theorem 1.** There exists a unique solution to the reconstruction problem $s_z = \tilde{\Phi} s_x$ with probability 1, if $K_f = ||s_x||_0 < \text{Spark}(\tilde{\Phi})/2 = (M + 1)/2$.

Theorem 1 guarantees the existence of a unique solution if the criteria $(M + 1)/2 > K_f$ is met with probability 1. However, the uniqueness criteria can still be met with $(M + 1)/2 < K_f$ with a non-zero probability. In the section below, we show that a unique solution to the reconstruction problem can be obtained with a quite high probability even when $(M + 1)/2 < K_f$, which is a typical case in the problem of reconstruction of the Nyquist SCF.

**C. Probabilistic Spark of the Sampling Matrix**

We have shown in the previous section that if $(M + 1)/2 > K_f$, then the resulting reconstruction method will result in a unique solution. Further, if $K_f > M^2$, then the resulting matrix $\tilde{\Phi} = \Phi_{\{1:k_f\}}$ will be rank deficient with probability 1 since $\text{Rank}(\tilde{\Phi}) = M^2$. In the regime where $(M + 1)/2 < K_f \leq M^2$, the resulting matrix can be full rank with a certain non-zero probability. Let $p_k$ be the probability that $\tilde{\Phi}$ is full column rank with $K_f = k, (M + 1)/2 < k \leq M^2$. The probability $p_k$ is NP-hard to compute since it requires us to find all the possible subsets of $k$ columns of $\tilde{\Phi}$ that are dependent, defined as Signature$(\tilde{\Phi})_k$ in [25]. Still, bounds on the Signature can be derived. One such interesting bound based on the known Spark is described in [26].

In the rest of this section, we find empirically the probability $p_k$ that the chosen matrix is full column rank, and hence will result in a unique reconstruction.

1. **Spark$(\tilde{\Phi})/2 \leq K_f < \text{Spark}(\tilde{\Phi})$: Theorem 3 in [25] gives a result for strong uniqueness of the reconstructed vector, showing that the probability to find an alternative representation with cardinality $K_f$ or smaller in the specified range is zero. Hence, if $M \geq K_f$, the reconstruction will result in the unique sparsest solution with probability 1.

2. **Spark$(\tilde{\Phi}) \leq K_f \leq \text{Rank}(\tilde{\Phi})$: From the definition of the Spark, there is a non-zero probability of at least one subset of $K_f$ columns being dependent where $K_f$ is in the range $M + 1 \leq K_f \leq M^2$. Here we focus on the maximum number of dependent sets of $K_f$ columns to calculate the probability $p_{K_f}$ that the chosen matrix is full-column rank.**

Fig.1 shows the probability $p_{K_f}$ that all possible subsets of $K_f$ columns of $\tilde{\Phi}$ are independent as a percentage of
In the previous simulation, there were no constraints placed on how to choose the $K_f$ columns from the $\hat{\Phi}$ matrix. But when operating in a low SNR regime, the PSD points in the Nyquist SCF are always non-zero with high values irrespective of the actual number of signals present. It is therefore imperative to reconstruct the PSD when operating at low SNRs. We add this condition while choosing the $K_f$ subsets of columns, i.e. instead of randomly selecting any set of columns, we ensure that the $N$ PSD columns are always a subset of the $K_f$ chosen columns and test for independence. Hence, $K_f$ is in the range $N \leq K_f \leq M^2$.

Let the system of equations, $s_z = \hat{\Phi}\hat{s}_z$, be overdetermined with $M^2 \geq K_f$. The reconstruction of the cyclic features of the SCF always exists and results in a unique solution with probability 1 iff $M \geq K_f$, or with probability $p_{K_f}$ iff $M + 1 \leq K_f \leq M^2$, and is given by

$$\hat{s}_z = \hat{\Phi}^+ s_z = [\hat{\Phi}^H \hat{\Phi}]^{-1} \hat{\Phi}^H s_z.$$  

### E. Signal Detection Using the Reconstructed SCF $\hat{s}_z$

The presence/absence of the signal is detected by collecting energies in the reconstructed non-zero points of the vectorized SCF $\hat{s}_z$. In this paper, we concentrate on a single-cycle detector which detects the energy of the most dominant peak present in the SCF, for e.g. the peak at $(2f_c, 0)$ for BPSK modulation. The test statistic is given by

$$T = |\hat{s}_z(i)|^2,$$

where $i$ corresponds to the index of the $(2f_c, 0)$ peak in the vectorized SCF. This statistic $T$ is then compared to a threshold $\gamma$ which is set according to the acceptable probability of false alarm in the system. The probability of false alarm and probability of detection can be expressed as $P_{fa} = P(T > \gamma|H_0)$, $P_d = P(T > \gamma|H_1)$ respectively.

In the next section, we relate the conditions for uniqueness of the reconstruction given by (17) to the compression ratio and the sparsity of the wideband channel.
V. MINIMUM LOSSLESS SAMPLING RATES AND ITS RELATION TO SCF SPARSITY LEVEL

In the compressive sensing literature, the question that often arises is how low can the compression ratio be for a given sparsity level enabling lossless reconstruction of the signal. By only exploiting the sparsity level in the PSD, the minimum compression ratio for a non-blind reconstruction (Landau rate) can be as low as the spectrum occupancy \( \alpha \), defined as the ratio between occupied bandwidth to the total channel bandwidth. Further, the PSD sparsity can only be exploited if we are operating in a high SNR region. At low SNRs, the noise power is high, the PSD is not sparse any more even if we are operating in a high SNR region. At low SNRs, as the ratio between occupied bandwidth to the total channel bandwidth can be as low as the spectrum occupancy \( \alpha \). Hence, the signal reconstruction will fail and we cannot reduce sampling rates even at low SNRs using only PSD sparsity.

The Landau rate, which we define as the minimum sampling rate based on the PSD sparsity is expressed as

\[
\left( \frac{M}{N} \right)_{Landau} = \frac{K_f \cdot \text{PSD}}{N} = 2Kf \frac{B}{f_{nyq}}.
\]

(19)

By operating in the cyclic domain, the sparsity has to account for non-zero cyclic frequencies as well, which yields a sparser transformation domain, and intuitively, the compression ratio can be reduced further. Since the noise is stationary, there are no features at non-zero cyclic frequencies which enables us to reduce sampling rates even at low SNRs. Fig. 3 shows the support of the SCF for a real bandlimited signal at high SNRs. Therefore, if we consider a sparse multiband signal \( x(t) \) in (1), its SCF is sparse in both \( \alpha \) and \( f \).

In the two-dimensional SCF, each bandlimited signal with bandwidth \( B \) occupies four lozenges with diagonals \( B \) and \( 2B \). Hence the total number of points \( K_f \) out of \( N^2 \) points of the SCF occupied by \( K \) signals at high SNRs (\( \geq 10 \text{dB} \)) is

\[
K_f = 4K \left( \frac{BN}{f_{nyq}} \right)^2
\]

(20)

At low SNRs (\(< 10 \text{dB}\)), we include points corresponding to the entire PSD in addition to the points due to the signals giving the following expression for \( K_f \)

\[
K_f = 4K \left( \frac{BN}{f_{nyq}} \right)^2 + N - 2K \left( \frac{BN}{f_{nyq}} \right)
\]

(21)

where \( N \) accounts for \( N \) PSD points in the SCF and the factor \( 2K(\frac{BN}{f_{nyq}}) \) accounts for the overlap between the total number of points occupied by \( K \) signals at high SNRs and the PSD.

From Theorem 2, we obtain a unique solution to the Nyquist SCF reconstruction with probability 1 iff \( M \geq K_f \), or with probability \( p_{K_f} \) iff \( M + 1 \leq K_f \leq M^2 \). For the case, \( M \geq K_f \), the minimum sampling ratio is given by \( \left( \frac{M}{N} \right)_{\text{min}} = \frac{K_f}{M} \). From (20) and (21), we have that \( K_f \sim N^2 \) and thus \( \frac{K_f}{N} \sim N \). Hence, with increasing \( N \), the resolution in \( f \) and \( \alpha \) increases, and the them the sampling ratio becomes greater than 1. This is mainly because the number of non-zero points \( K_f \) in the SCF is generally greater than \( M \).

We now focus on the range \( M + 1 \leq K_f \leq M^2 \) which guarantees a unique solution with probability \( p_{K_f} \). From section IV-B, we see that the probability \( p_{K_f} \) is a function of compression ratio and \( N \). To obtain a lower bound on the minimum sampling rate we study the case when \( K_f = \text{Rank}(\bar{\Phi}) = M^2 \) which is achieved with probability \( p_{M^2} \). This choice of \( K_f \) is not unrealistic as is shown in section IV-B where for moderate \( M \) and \( N \) \((M = 18, N = 36)\), \( p_{M^2} \) is greater than 0.98. Thus the lower bound is given by

\[
\left( \frac{M}{N} \right)_{cyclic} = \sqrt{\frac{K_f}{N^2}}
\]

(22)

\[
= \left\{ \begin{array}{ll}
2\sqrt{\frac{K}{N}} & \text{under high SNR} \\
\frac{2\sqrt{K}}{N} + \frac{1}{N} - \frac{2K}{N^2} & \text{under low SNR}
\end{array} \right.
\]

In (22), as \( N \rightarrow \infty \), for low SNR \( \left( \frac{M}{N} \right)_{cyclic} \rightarrow 2\sqrt{K} f_{nyq} \) which is same as the result for high SNR. This is intuitive because as \( N \rightarrow \infty \), the ratio \( \frac{N}{N} \rightarrow 0 \) and therefore the number of PSD points are negligible compared to the area of the SCF. Thus asymptotically the minimum sampling ratio is independent of SNR.

The minimum sampling rate for \( K_f = M(M + 1)/2 \) under a high SNR regime can similarly be expressed as

\[
\left( \frac{M}{N} \right)_{cyclic} = -1 + \frac{\sqrt{1 + 32K N^2 B^2 f_{nyq}^2}}{2N}.
\]

(23)

Comparing the expressions for minimum achievable sampling ratios in (19) and (22) corresponding to Energy and Cyclostationary detection respectively, it is evident that Cyclostationary detection
detectors require lower sampling rates for the same PSD sparsity levels. As this difference in minimum sampling ratios is \( \sim \sqrt{K} \), where \( K > 1 \), with increasing number of signals in the band/decreasing sparsity, Cyclostationary detectors have correspondingly increasing gains compared to Energy detectors.

VI. DISCUSSION AND NUMERICAL RESULTS

We consider a wideband channel of bandwidth 300 MHz, occupied with \( K \) BPSK signals one of which is the signal of interest (SOI). Each of the \( K \) signals has an effective signal bandwidth of 15 MHz (taking into account the roll-off factor of the pulse-shaping filter). As mentioned earlier, we assume an equal in-band SNR for all signals, and a frame length \( N = 36 \) samples. The spectral correlation peak considered for single-cycle detection is at \((\alpha, f) = (2f_c, 0)\). In Section VI-A, we compare the minimum compression ratios of the proposed detector to that of energy detection at low and high SNRs. In Section VI-B, the reconstruction error is shown in terms of the resulting MSE for various compression ratios. Section VI-C shows the relationship between the required sensing time for a given SNR and PSD sparsity to reach a desired point on the ROC curve. Finally, Section VI-D shows the effect of the sensing time, compression ratio, and SNR on the resulting ROC curves.

A. Theoretical Minimum Lossless Compression

Here, we compare the minimum lossless sampling rates obtained from the Landau rate in (19) and the lower bounds on sampling rates for Cyclostationary detectors corresponding to \( K_f = M^2 \) and \( M(M+1)/2 \) in (22) and (23). From Fig. 4(a), we observe that at high SNRs, when the PSD is fully occupied and therefore non-sparse, one cannot sample below the Nyquist rate. However, exploiting the 2D sparsity in \( f \) and \( \alpha \), we can go as low as \( M/N = 0.23 \) for \( K_f = M^2 \), giving about 80% lower sampling rates. For \( K_f = M(M+1)/2 \), we get a gain in compression ratios of about 70% compared to the Landau rate.

In Fig. 4(b), we study the minimum achievable sampling rates at low SNRs. In this scenario, with only PSD sparsity, we cannot go lower than the Nyquist rate irrespective of the actual number of signals present in the band of interest as the PSD is not sparse due to high noise power. However, we can exploit the sparsity in the cyclic frequency domain to go lower than the Nyquist rate. Fig. 4(b) shows that at low SNRs, with increasing \( N \), the minimum sampling rates for Cyclostationary detectors converges with the rates achievable at high SNRs, which acts as a lower bound for the case when \( K_f = M^2 \). But for moderate values of \( N \), for instance \( N = 36 \), we need about 10% higher sampling rates compared to the lower bound. Hence, there exists a tradeoff between the minimum lossless sampling rate and the reconstruction computational complexity which is also governed by \( N \). In fact, the lower bound on the minimum sampling rate can only be achieved as \( N \) grows, which entails additional computational complexity during reconstruction.

### B. SCF Reconstruction Mean Squared Error (MSE)

In this section, we consider noiseless signals, all of the same signal energy, and compute the mean squared error (MSE) of the cyclic feature used for detection at a compression ratio \( M/N \) with respect to its energy under no compression denoted by \( s_i \). We define the MSE as

\[
\text{MSE} = E \left[ \frac{|\hat{s}_x - s_i|^2}{|s_i|^2} \right] .
\]

where \( \hat{s}_x \) is the cyclic feature used for detection. We use the reconstruction algorithm with high \( L \) (of the order of \( 1 \times 10^4 \)) to ensure that the reconstructed \( \hat{s}_x \) is sparse with \( K_f \) non-zero spectral peaks, and show the loss in reconstructed cyclic feature with decreasing compression ratio. Fig. 5 shows the MSE versus compression ratio \( M/N \) for the channel populated with \( K = 1 \) and 3 signals, yielding a PSD occupancy of 5% and 15% respectively. The MSE curves show that...
there is no loss in spectral correlation peak energy up to a certain threshold compression ratio, below which the MSE starts increasing. Therefore, when $s_x$ is truly sparse (with sufficient $L$), the sampling rate can be reduced up to a certain threshold with no loss incurred, a point that we refer to as compression wall. All compression ratios above this threshold yield a lossless reconstruction. Fig. 5 shows this trend for different spectral occupancies, and the vertical lines point to the location of the corresponding compression walls. Note that the minimum lossless compression ratios obtained numerically in Fig. 5 are higher than the theoretical bounds computed using Eq. (22) due to spectral leakage of the signal at low $N$, which results in an effective number of non-zero features greater than (21).

C. Sensing Time Requirements and Comparison to Nyquist Detection

In this section, we quantify the additional sensing time required to reach a given $(P_{fa}, P_d) = (0.1, 0.9)$ at various SNRs and compression ratios using our reduced-complexity detector.

We start by comparing the sensing time (number of frames $L$) required to reach both Nyquist and sub-Nyquist based detectors for varying SNR and sparsity levels. Fig. 6 compares the Nyquist-based sensing time as a function of SNR to the required sensing time of sub-Nyquist based detector with compression ratio $M/N = 0.5$ with a channel populated with $K = 1$ and 3 signals. The result shows that the sub-Nyquist detection using our algorithm maintains the same relationship between the sensing time and SNR as the Nyquist detection. The slope of the sub-Nyquist curve is equal to that of the Nyquist detector showing that the reconstruction conserves the linear relationship (in dB scale) between SNR and sensing time for cyclostationary detectors.

Fig. 7 shows the trend of total number of samples/sensing time $(L \times M)$ versus the compression ratio under a fixed sparsity of $K = 1$ and 3 signals at SNR $= -5$ dB. As expected, the minimum lossless compression ratio $(M/N)_{\min}$ for the given sparsity is reached at about a compression ratio of 0.3 and 0.4 respectively as was also shown in the MSE plot in Fig. 5. With decreasing compression ratio, there is a steady increase in the required sensing time up to the compression wall. However, operating at a compression ratio below the compression wall requires an exponentially increasing number of samples $(L \times M)$ in order to operate at the same point on the ROC, which makes the detection infeasible within a constrained sensing time. This explains the tradeoff between the sensing time and reduction in sampling rates where being close to the compression wall is the optimum point to be at since the maximum savings in terms of sampling rates can be achieved with only a linear increase in sensing time.
D. Detection Performance Comparison

In this section we analyze the detection performance of the proposed algorithm for varying parameters. Firstly, we show the effect of varying the compression ratio under a fixed SNR = -10dB and $L = 15000$ averages for a fixed sparsity of 15% ($K = 3$ signals). Fig. 8 shows that decreasing the sampling rate does actually result in a gradual decay in detection performance. The reason behind this is that $L = 15000$ averages are not enough to ensure that the reconstructed $\hat{s}_x$ is truly sparse at a SNR of -10 dB, and therefore taking into account only $K_f$ spectral peaks according to Eq. (22) is not sufficient. Under enough averages, the MSE curve has shown that the reconstruction is lossless until the bound, and therefore the ROC curves will match with increasing $L$. The detection performance at sampling rates below the compression wall worsens drastically giving a linear relationship between $P_d$ and $P_{fa}$.

The presence of unwanted cyclic features at low SNRs in the reconstructed SCF can be compensated by increasing the number of frame averages $L$ needed to compute the vectorized auto-correlation matrix $r_z$. Fig. 9 shows the improvement in detection performance with increasing number of averages for fixed SNR = -10dB, compression ratio $M/N = 0.5$ and $K = 3$ signals. The detection performance improves with increasing $L$ as this corresponds to additional sensing time and also as the sparsity of the reconstructed SCF converges to its true sparsity.

Finally, we show the impact of varying SNR on detection performance for fixed $L = 15000$, compression ratio $M/N = 0.5$ and $K = 3$ signals. From Fig. 10 we can see that the detection performance improves with increasing SNR as 1) the signal power increases and 2) with increasing SNR the reconstructed SCF approaches the true sparsity. In fact, increasing SNR is equivalent to increasing the number of samples used for averaging the SCF, and the trend is similar to the ROC curves presented in Fig. 9.

VII. Conclusion

We have presented in this paper a compressive sampling approach for wideband spectrum sensing using cyclostationary detection which directly reconstructs the test statistic used for cyclostationary detection without reconstructing the time-domain signal. The resulting measurement matrix is shown to satisfy the RIP property with high probability, ensuring the stability of the reconstruction algorithm. Further, bounds on the minimum compression ratio are theoretically derived that ensure the uniqueness of the reconstructed statistic with high probability. The resulting compression bound is then compared to the theoretical Landau rate for energy detection, and gains between 73% and 83% in sampling rates are shown to be achieved under the proposed detector using moderate frame length, and hence without incurring high computational complexity. With respect to the detection performance resulting from the unique stable reconstruction, the relationship between SNR and sensing time remains linear on a logarithmic scale as for Nyquist detectors. Finally, it was shown that under
the minimum compression ratio, the reconstruction becomes non-unique, which results in detection performance loss which could be made up for with exponentially increasing sensing time.

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